No Need To Be Strict: on the expressiveness of metric temporal logics with (non-)strict operators

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Abstract

The *until* modality of temporal logic "*A* until *B*" is called *strict* in its first argument when it does not constrain the value of the first argument *A* at the instant at which the formula is evaluated. In this paper, we show that linear metric temporal logics with bounded *until* modalities that are non-strict in the first argument are as expressive as those with strict modalities, when interpreted over non-Zeno dense-time models.

1 Introduction

Propositional Linear Temporal Logic (LTL) is a well-established formalism for describing and reasoning about systems that evolve over time. On the one hand, its success depends on the temporal modalities — the basic ingredient of temporal logic — mirroring naturally and effectively the way humans reason intuitively about time. On a more technical level, the classical theory of temporal logic over discrete time has been thoroughly studied and its deep connections with other theories (such as automata theory and monadic second-order logic) are solid and well-understood [5].

Matters change considerably when trying to extend this robust theory to deal with dense time and metric (i.e., quantitative) constraints. Achieving a unifying theory has proved much harder in this case. In particular, results are often much *less robust* when dealing with dense metric time: small changes in the definition of basic operators, or in the choice of — apparently marginal — semantic details, may yield considerable changes in the expressiveness of the resulting formalism. As a witness to such claims, consider for instance the recent results about the expressiveness and decidability of Metric Temporal Logic (MTL) [6, 20, 21].

There have been several suggestions for the causes of, and solutions to, this lack of robustness; see for instance [5, 16, 2]. This highlights the necessity for a close scrutiny of the variations in the definition of temporal logic languages, in order to assess precisely their actual impact on expressiveness and other issues.

In this paper, we carry out such an analysis about the issue of *strictness*, with reference to the language MTL over dense-time models. MTL is built upon a single parametric binary modality U_I named *bounded until*. Informally, $\mathsf{U}_I(\mathsf{p}, \mathsf{q})$ means that p holds from the current instant until q holds sometimes in the future, within the time interval I. If we choose an *until strict* in its first argument, then p is not required to hold exactly at the current instant (i.e., the *until* operator constraints strictly the future); otherwise, with a *non-strict until*, p has to hold at the current instant as well.

Although, to the best of our knowledge, the issue of strictness has never been investigated in detail for metric dense-time models, the usual informal assumption is that strict operators are more expressive than non-strict ones (see for instance [1, 14, 6]). In this paper we analyze the problem for non-Zeno dense-time models, by showing that in this case MTL with strict *until* (in the first argument) is exactly as expressive as MTL with non-strict *until*.

Outline. The paper is organized as follows. Section 2 presents MTL by introducing its strict and non-strict variants; moreover, it references other metric temporal logics that are similar to MTL, and to which (some of) the results of this paper apply. Section 3 presents simple results on the relationship between strict and non-strict operators; in particular, it recalls the straightforward results that strict operators are at least as expressive as non-strict ones, and it proves the equivalence — over non-Zeno behaviors — between the strict and non-strict variants of the derived operator *nowon*. Section 4 proves the main result, that is that MTL with *until* strict in the first argument is exactly as expressive as MTL with non-strict *until*. Finally, Section 5 discusses, in a partly informal way, to what extent the results of the paper can be applied to other common semantic models; in particular, it considers the use of past operators, of mono-infinite time domains, and of generic (Zeno) models.

2 Metric Temporal Logic(s)

2.1 MTL

In this section, we define the syntax (Section 2.1.1) and semantics (Section 2.1.2) of MTL. Although full MTL, as defined originally by Koymans [18], is a very expressive first-order language, in this paper we refer to MTL *à la* Alur and Henzinger [3], that is to a propositional fragment of the full language. This distinction is often made only implicitly in the literature.

2.1.1 Syntax

Non-strict MTL (denoted simply as MTL in this paper) is defined by the following syntax, where $I = \langle a, b \rangle$ denotes a non-empty[1] interval of the time domain $\mathbb{T}$ such that $0 \leq a \leq b \leq +\infty$, $\langle$ is (or [, $\rangle$ is) or], and $\mathsf{p} \in \mathcal{P}$ is some atomic proposition from a finite set $\mathcal{P}$.

$$\phi \quad ::= \quad \mathsf{p} \mid \mathsf{U}_I(\phi_1, \phi_2) \mid \neg\phi \mid \phi_1 \wedge \phi_2$$

As it will be apparent in the definition of the semantics, we mean U_I to denote a bounded *until* that is non-strict in its first argument.[2]

Strict MTL. Strict MTL (denoted as $\widetilde{\text{MTL}}$) is obtained by replacing the non-strict bounded *until* U_I with its strict variant, denoted as $\widetilde{\mathsf{U}}_I$. The overall goal of this paper is to show that MTL is exactly as expressive as $\widetilde{\text{MTL}}$, when interpreted over non-Zeno models.

Derived operators. Besides the usual abbreviations for propositional connectives and constants (such as $\vee$, $\Rightarrow$, and $\top$), it is customary to define a set of derived temporal operators. Table 1 lists those that are referenced in this paper, together with their definitions. Each operator is defined in both its non-strict version (on the left-hand column) and its strict version (on the right-hand column); the latter is denoted graphically by a tilde. We also denote *punctual* intervals (i.e., intervals of the form $[d, d]$ for some d) with the abbreviation $= d$.

2.1.2 Semantics

We define formally the semantics of MTL (and $\widetilde{\text{MTL}}$) over generic Boolean behaviors. Given a time domain $\mathbb{T}$ (satisfying some standard minimal properties [18]) and a finite set of atomic propositions $\mathcal{P}$, a *Boolean behavior* over $\mathcal{P}$ is a mapping $b : \mathbb{T} \to 2^{\mathcal{P}}$ from the time domain to subsets of $\mathcal{P}$: for every time instant $t \in \mathbb{T}$, b maps t to the set of propositions $b(t)$ that are true at t. We denote the set of all mappings for a given set $\mathcal{P}$ as $\mathcal{B}_{\mathcal{P}}$, or simply as $\mathcal{B}$.

The semantics of MTL (and $\widetilde{\text{MTL}}$) formulas is given through a satisfaction relation $\models_{\mathbb{T}}$: given a behavior $b \in \mathcal{B}$, an instant $t \in \mathbb{T}$ (sometimes called "current

[1]That is there exists some $p \in I$; we rule out empty intervals for simplicity of presentation.
[2]Non-strict operators are also called *reflexive* [8].

Operator	$\equiv$	Definition	Operator	$\equiv$	Definition
$\Diamond_I(\phi)$	$\equiv$	$\mathsf{U}_I(\top, \phi)$	$\widetilde{\Diamond}_I(\phi)$	$\equiv$	$\widetilde{\mathsf{U}}_I(\top, \phi)$
$\Box_I(\phi)$	$\equiv$	$\neg\Diamond_I(\neg\phi)$	$\widetilde{\Box}_I(\phi)$	$\equiv$	$\neg\widetilde{\Diamond}_I(\neg\phi)$
$\mathsf{U}^{\downarrow}_I(\phi_1, \phi_2)$	$\equiv$	$\mathsf{U}_I(\phi_1, \phi_2 \wedge \phi_1)$	$\widetilde{\mathsf{U}}^{\downarrow}_I(\phi_1, \phi_2)$	$\equiv$	$\widetilde{\mathsf{U}}_I(\phi_1, \phi_2 \wedge \phi_1)$
$\mathsf{U}(\phi_1, \phi_2)$	$\equiv$	$\mathsf{U}_{(0,+\infty)}(\phi_1, \phi_2)$	$\widetilde{\mathsf{U}}(\phi_1, \phi_2)$	$\equiv$	$\widetilde{\mathsf{U}}_{(0,+\infty)}(\phi_1, \phi_2)$
$\mathsf{U}^{\mathrm{w}}(\phi_1, \phi_2)$	$\equiv$	$\mathsf{U}_{[0,+\infty)}(\phi_1, \phi_2)$	$\widetilde{\mathsf{U}}^{\mathrm{w}}(\phi_1, \phi_2)$	$\equiv$	$\widetilde{\mathsf{U}}_{[0,+\infty)}(\phi_1, \phi_2)$
$\mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)$	$\equiv$	$\mathsf{U}^{\mathrm{w}}(\phi_1, \phi_2 \wedge \phi_1)$	$\widetilde{\mathsf{U}}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)$	$\equiv$	$\widetilde{\mathsf{U}}^{\mathrm{w}}(\phi_1, \phi_2 \wedge \phi_1)$
$\bigcirc(\phi)$	$\equiv$	$\mathsf{U}(\phi, \top)$	$\widetilde{\bigcirc}(\phi)$	$\equiv$	$\widetilde{\mathsf{U}}(\phi, \top)$

Table 1: MTL derived temporal operators

instant") and an MTL (or $\widetilde{\text{MTL}}$) formula ϕ, the satisfaction relation is defined inductively as follows.

$b(t) \models_{\mathbb{T}} \mathsf{p}$	iff	$\mathsf{p} \in b(t)$
$b(t) \models_{\mathbb{T}} \mathsf{U}_I(\phi_1, \phi_2)$	iff	there exists $d \in I$ such that $b(t + d) \models_{\mathbb{T}} \phi_2$ and, for all $u \in [0, d)$ it is $b(t + u) \models_{\mathbb{T}} \phi_1$
$b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_I(\phi_1, \phi_2)$	iff	there exists $d \in I$ such that $b(t + d) \models_{\mathbb{T}} \phi_2$ and, for all $u \in (0, d)$ it is $b(t + u) \models_{\mathbb{T}} \phi_1$
$b(t) \models_{\mathbb{T}} \neg\phi$	iff	$b(t) \not\models_{\mathbb{T}} \phi$
$b(t) \models_{\mathbb{T}} \phi_1 \wedge \phi_2$	iff	$b(t) \models_{\mathbb{T}} \phi_1$ and $b(t) \models_{\mathbb{T}} \phi_2$
$b \models_{\mathbb{T}} \phi$	iff	for all $t \in \mathbb{T}$: $b(t) \models_{\mathbb{T}} \phi$

Notice that the above definition works both with mono-infinite (e.g., $\mathbb{N}, \mathbb{R}_{\geq 0}, \mathbb{Q}_{\geq 0}$) and with bi-infinite time domains (e.g., $\mathbb{Z}, \mathbb{R}, \mathbb{Q}$). Still, it differs from the most common definition for mono-infinite time domains, where *initial satisfiability* is usually chosen: see Section 5, where we discuss why our choice is compatible with the more common definition for mono-infinite time domains.

Now, according to the semantics we have just shown, let us make a few remarks on the definitions of Table 1.

- Since we consider strictness in the first argument only, it is obvious that the strict and non-strict variants of the $\Diamond$ and $\Box$ operators have the same semantics.

- $\mathsf{U}^{\downarrow}_I$ denotes the *matching* variant of the *until* operator, where both arguments are required to hold together at some point. This is a relatively uncommon variant, although it has been used in works such as [9, 19]. We deal with it explicitly as its use simplifies the presentation of some equivalences in Section 4.3. We call the standard variant *non-matching until*.

- U denotes the *qualitative until*, the basic operator of classic temporal logic.

U^{w} denotes instead the *weak* version of the qualitative *until*, where the second argument may hold at the current instant as well; this variant is sometimes called *non-strict in the second argument*.[3]

- $\bigcirc$ denotes what we call the *nowon* operator. Informally, for dense-time models, $\bigcirc(\phi)$ means that ϕ holds continuously on some non-empty — but arbitrarily small — interval to the right of the current instant.

2.1.3 Non-Zenoness

It is common to constrain behaviors over dense time to the *non-Zenoness* (also called *finite variability*) requirement [4, 12]. A behavior $b \in \mathcal{B}$ is called *non-Zeno* if the truth value of any atomic proposition $\mathsf{p} \in \mathcal{P}$ changes in b only finitely many times over any bounded interval of time. Formally, we require that for any $t \in \mathbb{T}$ (where $\mathbb{T}$ is assumed to be dense) there exists an $\epsilon > 0$ such that, for all $\mathsf{p} \in \mathcal{P}$: (1) either $\mathsf{p} \in b(u)$ for all $u \in (t, t+\epsilon)$ or $\mathsf{p} \notin b(u)$ for all $u \in (t, t+\epsilon)$; and (2) either $\mathsf{p} \in b(v)$ for all $v \in (t-\epsilon, t) \cap \mathbb{T}$ or $\mathsf{p} \notin b(v)$ for all $v \in (t-\epsilon, t) \cap \mathbb{T}$. In particular, it can be easily seen that $b \models_{\mathbb{R}} \Box_{[0,+\infty)}\big(\widetilde{\bigcirc}(\mathsf{p}) \vee \widetilde{\bigcirc}(\neg\mathsf{p})\big)$ must hold for any non-Zeno behavior b and atomic proposition p.

This derived property can be "lifted" from atomic propositions to generic MTL formulas, by observing that MTL operators preserve non-Zenoness (the proof is straightforward by structural induction). Then, for any formula ϕ and non-Zeno behavior b:

$$b \models_{\mathbb{T}} \Box_{[0,+\infty)}\big(\widetilde{\bigcirc}(\phi) \vee \widetilde{\bigcirc}(\neg\phi)\big) \tag{1}$$

Throughout this paper, we consider only dense-time behaviors that are non-Zeno. At the end of Section 5 we discuss the impact of allowing Zeno behaviors.

2.2 Other Metric Temporal Logics

Although in this paper we consider MTL, there are several other metric temporal logic languages that are closely related to MTL, to which our results apply as well. In particular, let us mention:

- ${}^{\mathbb{R}}_{\mathbb{Z}}$TRIO, a propositional subset of the TRIO language [13, 7]. ${}^{\mathbb{R}}_{\mathbb{Z}}$TRIO has been introduced in [9], and the results of the present paper have been first developed for ${}^{\mathbb{R}}_{\mathbb{Z}}$TRIO in [10].

- MITL, a decidable subset of the MTL language introduced by Alur, Feder, and Henzinger in [1]. MITL is the subset of the MTL language where all

[3]On the other hand, the term *weak* sometimes denotes a different variation of the *until*, namely one where the second argument is not required to hold eventually [8].

intervals are required to be *non-punctual*. In this paper, we name MITL the non-punctual subset of non-strict MTL, and $\widetilde{\text{MITL}}$ the non-punctual subset of strict $\widetilde{\text{MTL}}$.

3 Preliminary and Auxiliary Results

This section presents a set of results that are auxiliary to the proof of the main result of equivalence, carried out in Section 4. For brevity, we omit the proofs of the simplest results.

Strict is at least as expressive as non-strict. First of all, it is straightforward to show that $\widetilde{\text{MTL}}$ is at least as expressive as its non-strict variant MTL. In fact, we have the following equivalence, which holds for both discrete and dense-time domains.

$$\mathsf{U}_I(\phi_1, \phi_2) \equiv \begin{cases} \phi_2 \vee (\phi_1 \wedge \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2)) & \text{if } I = [0, b\rangle \\ \phi_1 \wedge \widetilde{\mathsf{U}}_I(\phi_1, \phi_2) & \text{otherwise} \end{cases} \tag{2}$$

Over discrete time with metric, strict is as expressive as non-strict. If we consider discrete-time domains only (i.e., in practice $\mathbb{N}$, $\mathbb{Z}$, or subsets thereof), it is not difficult to express strict *until* using the non-strict version. In fact, we have the following equivalence *for discrete-time domains* (recall that, in discrete time, all intervals can be expressed as closed).

$$\widetilde{\mathsf{U}}_{[a,b]}(\phi_1, \phi_2) \equiv \begin{cases} \Diamond_{=1}\big(\mathsf{U}_{[a-1,b-1]}(\phi_1, \phi_2)\big) & \text{if } a \geq 1 \\ \phi_2 \vee \Diamond_{=1}\big(\mathsf{U}_{[a,b-1]}(\phi_1, \phi_2)\big) & \text{if } a = 0 < b \\ \phi_2 & \text{if } a = b = 0 \end{cases}$$

The equivalence in expressive power over discrete time does not contradict the well-known result that, in Linear Temporal Logic (LTL), strict and non-strict *until* are not equivalent, as the result for discrete-time LTL deals with qualitative (unbounded) *until* only, not with its quantitative (bounded) version. In this regard, let us recall that, in LTL, *next* [8] can be defined from strict *until*, but not from non-strict *until* only [17, 11]. On the contrary, it is easy to express the $\Diamond_{=1}$ operator (i.e., *next*) using only quantitative non-strict *until*.

In the remainder of the paper, we therefore assume the time domain $\mathbb{T}$ to be a dense set; in practice, we consider the sets $\mathbb{Q}$ and $\mathbb{R}$, and their mono-infinite restrictions $\mathbb{Q}_{\geq 0}$ and $\mathbb{R}_{\geq 0}$.

Qualitatively, weak non-strict is less expressive than strong strict. If we consider the qualitative versions of the *until* operator over generic dense-time behaviors, Reynolds [23] has shown that the strict and strong *until* operator $\widetilde{\mathsf{U}}$ is strictly more expressive than its non-strict and weak variant U^{w} (see also [8]).

Even if [23] deals with dense time as well as we do, let us remark that two assumptions are introduced in [23] that make it possible the separation of expressiveness: (1) only qualitative operators are considered; and (2) the strict *until* is compared against its non-strict *and* weak counterpart (or, equivalently, *until* strict in both arguments is compared against *until* strict in neither). On the contrary, in this paper we consider quantitative operators, and we focus on the issue of strictness (in the first argument) only, without considering the impact of weak variations.

For non-Zeno behaviors, non-strict *nowon* is as expressive as strict *nowon*. Let us prove the equivalence in expressive power of the strict and non-strict versions of the *nowon* operator $\bigcirc$. We establish the following equivalence, for non-Zeno behaviors.

$$\widetilde{\bigcirc}(\phi) \;\equiv\; \bigcirc(\phi) \vee (\neg\phi \wedge \neg \bigcirc(\neg\phi)) \tag{3}$$

Proof of Formula 3. It is clear that $\bigcirc(\phi) \equiv \phi \wedge \widetilde{\bigcirc}(\phi)$. Then, let us start by proving the $\Rightarrow$ direction: assume that $b(t) \models_{\mathbb{T}} \widetilde{\bigcirc}(\phi)$ holds at some instant t, for a non-Zeno behavior $b \in \mathcal{B}$. Let us first consider the case: ϕ true at t; then also $\bigcirc(\phi)$, and we satisfy the first term of the disjunction. Otherwise, ϕ is false at t. Then, $\bigcirc(\neg\phi)$ must also be false at t, otherwise $\widetilde{\bigcirc}(\phi)$ cannot be true. Thus both conjuncts of the second term of the disjunction are satisfied.

For the $\Leftarrow$ direction, let us start by considering the case $\bigcirc(\phi)$ at t; then, *a fortiori*, $\widetilde{\bigcirc}(\phi)$ at t and we are done. Otherwise, let us assume that $\neg\phi \wedge \neg \bigcirc(\neg\phi)$ holds at the current instant t. Since b is non-Zeno, we consider Formula 1 at t to immediately conclude that $\widetilde{\bigcirc}(\phi)$ at t. □

4 Strict Is As Expressive As Non-Strict

In this section, we establish the main result of the paper, that is that MTL is as expressive as $\widetilde{\text{MTL}}$ for non-Zeno behaviors. Since Formula 2 states that $\widetilde{\text{MTL}}$ is at least as expressive as MTL, the goal is now to prove the converse. In other words, we show how to express any occurrence of $\widetilde{\mathsf{U}}_{\langle a,b\rangle}$ in terms of non-strict *until* only. This is done by analyzing different cases for the left-bound a.

4.1 Case $a > 0$

First, we consider intervals I in $\widetilde{\mathsf{U}}_I$ with a positive left end-point.

Left-open intervals. For $a > 0$, we establish the following equivalence.

$$\widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2) \equiv \Diamond_{(a,b\rangle}(\phi_2) \wedge \Box_{(0,a]}(\mathsf{U}(\phi_1, \phi_2)) \tag{4}$$

Proof of Formula (4). Let us start with the $\Rightarrow$ direction: assume that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2)$. That is, there exists a $u \in (t + a, t + b\rangle$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $v \in (t, u)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. From $b(u) \models_{\mathbb{T}} \phi_2$ it follows immediately that $\Diamond_{(a,b\rangle}(\phi_2)$, so the first conjunct is proved.

Then, let us show that $b(t) \models_{\mathbb{T}} \Box_{(0,a]}(\mathsf{U}(\phi_1, \phi_2))$. Let α be any instant in $(0, a]$; we have to show that $b(t + \alpha) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$. Notice that $(t + a, t + b\rangle \subseteq (t + \alpha, +\infty)$, as $t + a \geq t + \alpha$, therefore $u \in (t + \alpha, +\infty)$ *a fortiori*. Moreover, $[t + \alpha, u) \subset (t, u)$, as $\alpha > 0$, so for all $v' \in [t + \alpha, u)$ it is $b(v') \models_{\mathbb{T}} \phi_1$. Therefore, we have shown that $b(t + \alpha) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$.

Let us now consider the $\Leftarrow$ direction. Notice that $b(t) \models_{\mathbb{T}} \Box_{(0,a]}(\mathsf{U}(\phi_1, \phi_2))$ implies that $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\phi_1)$. Moreover, in particular $b(t + a) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$. That is, there exists a $u \in (t + a, +\infty)$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $v \in [t + a, u)$ it is $b(v) \models_{\mathbb{T}} \phi_1$.

Let us now consider the case $u \in (t + a, t + b\rangle$. All in all, ϕ_1 holds over the interval (t, u), and ϕ_2 holds at u; therefore, we have $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2)$.

Otherwise, let us consider the case $u \notin (t + a, t + b\rangle$; therefore $u \in (t + a, +\infty) \setminus (t + a, t + b\rangle$. In particular, this implies that for all $v \in (t, t + b)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. Moreover, we are also assuming that $b(t) \models_{\mathbb{T}} \Diamond_{(a,b\rangle}(\phi_2)$ in this branch of the proof. That is, there exists a $u' \in (t+a, t+b\rangle$ such that $b(u') \models_{\mathbb{T}} \phi_2$. Since $(t, u') \subseteq (t, t+b)$, then we have shown that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2)$, as required. □

Left-closed intervals. The case for left-closed intervals is based on the previous one. In fact, we prove the following formula (still for $a > 0$), which relies on Formula 4 to replace strict occurrences of the *until* operator.

$$\widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2) \equiv \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2) \vee \left(\Box_{(0,a)}(\phi_1) \wedge \Diamond_{=a}(\phi_2)\right) \tag{5}$$

Proof of Formula (5). Beginning with the $\Rightarrow$ direction, assume that there exists a $u \in [t + a, t + b\rangle$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and for all $v \in (t, u)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. Then, if $u \in (t + a, t + b\rangle$, obviously $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2)$. Otherwise, it is $u = t + a$; therefore $b(t) \models_{\mathbb{T}} \Diamond_{=a}(\phi_2)$ and $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\phi_1)$.

For the $\Leftarrow$ direction, let us first consider $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(a,b\rangle}(\phi_1, \phi_2)$; then there exists a $u \in (t + a, t + b\rangle$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and for all $v \in (t, u)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. Since $(t + a, t + b\rangle \subset [t + a, t + b\rangle$, then *a fortiori* $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2)$. Otherwise, $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\phi_1) \wedge \Diamond_{=a}(\phi_2)$ implies $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2)$ for $u = t + a$. □

4.2 Case $a = 0$

Now, we handle the case of intervals $I = \langle a, b\rangle$ where $a = 0$.

Left-open intervals. The following equivalence, which relies on Formula 3 to replace strict occurrences of the *nowon* operator (and thus it assumes non-Zenoness), holds.

$$\widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2) \;\equiv\; \Diamond_{(0,b\rangle}(\phi_2) \wedge \widetilde{\bigcirc}(\mathsf{U}(\phi_1, \phi_2)) \tag{6}$$

Proof of Formula (6). Let us start with the $\Rightarrow$ direction: assume that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2)$. That is, there exists a $u \in (t, t + b\rangle$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $v \in (t, u)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. From $b(u) \models_{\mathbb{T}} \phi_2$ it follows immediately that $\Diamond_{(0,b\rangle}(\phi_2)$, so the first conjunct is proved.

Then, let us show that $b(t) \models_{\mathbb{T}} \widetilde{\bigcirc}(\mathsf{U}(\phi_1, \phi_2))$. More precisely, we can show that for all $\alpha \in (t, u)$ it is $b(\alpha) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$. In fact, notice that $u > \alpha$, so $u \in (\alpha, +\infty)$; moreover $[\alpha, u) \subset (t, u)$, as $\alpha > t$. All in all, we have that $b(\alpha) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$.

Let us now consider the $\Leftarrow$ direction, and let us assume the second conjunct: $b(t) \models_{\mathbb{T}} \widetilde{\bigcirc}(\mathsf{U}(\phi_1, \phi_2))$. That is, there exists a $\beta > 0$ such that for all $v \in (t, t + \beta)$ it is $b(v) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$. This implies that for all $v \in (t, t + \beta)$ it is also $b(v) \models_{\mathbb{T}} \phi_1$.

Let $v' = t + \beta/2$; since $v' \in (t, t + \beta)$ then $b(v') \models_{\mathbb{T}} \mathsf{U}(\phi_1, \phi_2)$. Thus, there exists a $u \in (t + \beta/2, +\infty)$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $z \in [t + \beta/2, u)$ it is $b(z) \models_{\mathbb{T}} \phi_1$. Notice that ϕ_1 holds throughout (t, u).

If $u \in (t, t + b\rangle$ then we are done proving $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2)$. Otherwise, $u \in (t, +\infty) \setminus (t, t + b\rangle$; then ϕ_1 holds throughout $(t, t + b\rangle \subset (t, u)$. Since in this branch of the proof we are also assuming that $b(t) \models_{\mathbb{T}} \Diamond_{(0,b\rangle}(\phi_2)$, we are done in this case as well. □

Left-closed intervals. It is immediate to prove the following equivalence, for $b > 0$.

$$\widetilde{\mathsf{U}}_{[0,b\rangle}(\phi_1, \phi_2) \;\equiv\; \widetilde{\mathsf{U}}_{(0,b\rangle}(\phi_1, \phi_2) \vee \phi_2 \tag{7}$$

For $a = b = 0$, it is trivial to verify the following.

$$\widetilde{\mathsf{U}}_{[0,0]}(\phi_1, \phi_2) \;\equiv\; \phi_2 \tag{8}$$

4.3 No Need for Punctual Intervals

In our set of equivalences, Formula 5 introduces a *punctual* interval, i.e., it specifies an exact time distance through the $\Diamond_{=a}$ operator. Therefore, it transforms $\widetilde{\text{MITL}}$ formulas (whenever $b > a$) into generic MTL formulas. This is not necessary, as far as the expression of the strict *until* is concerned. In fact, we now provide other equivalences — alternative to Formula 5 — that do not use punctual intervals (of course, provided the strict *until* itself is constrained by a non-punctual interval). Thus, they transform $\widetilde{\text{MITL}}$ formulas into equivalent MITL formulas.

Matching *until*. To this end, it is convenient to start with the *matching* variant of the *until* operator: we establish the following, for $a > 0$.

$$\widetilde{\mathsf{U}}^{\downarrow}_{[a,b\rangle}(\phi_1, \phi_2) \;\equiv\; \Box_{(0,a]}\Big(\mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)\Big) \wedge \Diamond_{[a,b\rangle}(\phi_2 \wedge \phi_1) \tag{9}$$

Proof of Formula 9. Let us start with the $\Rightarrow$ direction, and let t be the current instant. We assume that there exists a $u \in [t + a, t + b\rangle$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $v \in (t, u]$ it is $b(v) \models_{\mathbb{T}} \phi_1$. Clearly, $b(t) \models_{\mathbb{T}} \Diamond_{[a,b\rangle}(\phi_2 \wedge \phi_1)$ is immediately implied. We still have to show that, for all $d \in (t, t + a]$, it is $b(d) \models_{\mathbb{T}} \mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)$.

So, let us consider a generic d; notice that $u \in [d, +\infty)$ as $d \leq t + a \leq u$, and recall that $b(u) \models_{\mathbb{T}} \phi_2$. Moreover, let v' be any point in $[d, u]$; since $d > t$, *a fortiori* $v' \in (t, u]$. Thus, by hypothesis $b(v') \models_{\mathbb{T}} \phi_1$, so we are done with proving $b(d) \models_{\mathbb{T}} \mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)$.

Let us now consider the $\Leftarrow$ direction. First of all, let us realize that $b(t) \models_{\mathbb{T}} \Box_{(0,a]}\Big(\mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)\Big)$ implies $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\phi_1)$. In fact, otherwise there would be a $y \in (t, t + a)$ such that $b(y) \models_{\mathbb{T}} \neg\phi_1$; but since it is also $b(y) \models_{\mathbb{T}} \mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)$ we clearly have a contradiction.

Next, let us consider the consequences of $b(t + a) \models_{\mathbb{T}} \mathsf{U}^{\mathrm{w}\downarrow}(\phi_1, \phi_2)$: it means that there exists a $u' \in [t + a, +\infty)$ such that $b(u') \models_{\mathbb{T}} \phi_2$ and for all $v' \in [t + a, u']$ it is $b(v') \models_{\mathbb{T}} \phi_1$.

Let us first distinguish the case $u' \in [t + a, t + b\rangle$. All in all, ϕ_1 holds over the interval $(t, u']$, and ϕ_2 holds at u'; therefore, we have $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}^{\downarrow}_{[a,b\rangle}(\phi_1, \phi_2)$.

Otherwise, let us consider the case $u' \notin [t+a, t+b\rangle$; therefore $u' \in [t+a, +\infty) \setminus [t + a, t + b\rangle$. In particular, this implies that for all $v' \in (t, t + b)$ it is $b(v') \models_{\mathbb{T}} \phi_1$. Moreover, we are also assuming that $b(t) \models_{\mathbb{T}} \Diamond_{[a,b\rangle}(\phi_2 \wedge \phi_1)$ in this branch of the proof. That is, there exists a $u'' \in [t + a, t + b\rangle$ such that $b(u'') \models_{\mathbb{T}} \phi_2 \wedge \phi_1$. Since $(t, u'') \subseteq (t, t + b)$, then we have shown that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}^{\downarrow}_{[a,b\rangle}(\phi_1, \phi_2)$, as required. □

Non-matching *until*. Then, still for $a > 0$, we can express the non-matching variant by means of the matching variant.

$$\widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2) \;\equiv\; \widetilde{\mathsf{U}}^{\downarrow}_{[a,b\rangle}(\phi_1, \phi_2) \vee \Big(\Box_{(0,a)}(\mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2)) \wedge \Diamond_{[a,b\rangle}(\neg\phi_1 \wedge \phi_2)\Big) \tag{10}$$

Proof of Formula 10. Let us start with the $\Rightarrow$ direction, and let t be the current instant. We assume that there exists a $u \in [t + a, t + b\rangle$ such that $b(u) \models_{\mathbb{T}} \phi_2$

and, for all $v \in (t, u)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. If, furthermore, $b(u) \models_{\mathbb{T}} \phi_1$, then $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}^{\downarrow}_{[a,b\rangle}(\phi_1, \phi_2)$ and we are done. Otherwise, let us consider the case $b(u) \models_{\mathbb{T}} \neg\phi_1$: clearly, $b(t) \models_{\mathbb{T}} \Diamond_{[a,b\rangle}(\neg\phi_1 \wedge \phi_2)$ is immediately implied (as $\neg\phi_1 \wedge \phi_2$ holds at u). We still have to show that, for all $d \in (t, t + a)$, it is $b(d) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2)$.

So, let us consider a generic d; notice that $u \in (d, +\infty)$ as $d < t + a \leq u$, and recall that $b(u) \models_{\mathbb{T}} \phi_2 \wedge \neg\phi_1$. Moreover, let v' be any point in $[d, u)$; since $d > t$, *a fortiori* $v' \in (t, u)$. Thus, by hypothesis $b(v') \models_{\mathbb{T}} \phi_1$, so we are done with proving $b(d) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2)$.

Let us now consider the $\Leftarrow$ direction. The implication $\widetilde{\mathsf{U}}^{\downarrow}_{[a,b\rangle}(\phi_1, \phi_2) \Rightarrow \widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2)$ is straightforward, so let us assume that $b(t) \models_{\mathbb{T}} \Diamond_{[a,b\rangle}(\neg\phi_1 \wedge \phi_2)$ and $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2))$.

First of all, let us realize that $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2))$ implies $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\phi_1)$. In fact, otherwise there would be a $y \in (t, t + a)$ such that $b(y) \models_{\mathbb{T}} \neg\phi_1$; but since it is also $b(y) \models_{\mathbb{T}} \mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2)$ we clearly have a contradiction, as the latter implies in particular that $b(y) \models_{\mathbb{T}} \phi_1$.

Next, let us realize that $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\phi_1)$ and $b(t) \models_{\mathbb{T}} \Box_{(0,a)}(\mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2))$ holding together require that $b(t + a) \models_{\mathbb{T}} \mathsf{U}^{\mathrm{w}}(\phi_1, \neg\phi_1 \wedge \phi_2)$. In fact, ϕ_1 cannot become false before $t + a$, but it *must* become false somewhere after (or at) $t + a$ to satisfy $\mathsf{U}(\phi_1, \neg\phi_1 \wedge \phi_2)$. Thus, there exists a $u' \in [t + a, +\infty)$ such that $b(u') \models_{\mathbb{T}} \neg\phi_1 \wedge \phi_2$ and for all $v' \in [t + a, u')$ it is $b(v') \models_{\mathbb{T}} \phi_1$.

Let us first distinguish the case $u' \in [t + a, t + b\rangle$. All in all, ϕ_1 holds over the interval (t, u'), and ϕ_2 holds at u'; therefore, we have $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2)$.

Otherwise, let us consider the case $u' \notin [t+a, t+b\rangle$; therefore $u' \in [t+a, +\infty) \setminus [t + a, t + b\rangle$. In particular, this implies that for all $v' \in (t, t + b)$ it is $b(v') \models_{\mathbb{T}} \phi_1$. Moreover, we are also assuming that $b(t) \models_{\mathbb{T}} \Diamond_{[a,b\rangle}(\neg\phi_1 \wedge \phi_2)$ in this branch of the proof. That is, there exists a $u'' \in [t + a, t + b\rangle$ such that $b(u'') \models_{\mathbb{T}} \neg\phi_1 \wedge \phi_2$. Since $(t, u'') \subseteq (t, t + b)$, then we have shown that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{U}}_{[a,b\rangle}(\phi_1, \phi_2)$, as required. □

4.4 Summary

Considering non-Zeno behaviors, Formulas 4 through 8 provide a way to replace any occurrence of strict *until* with an equivalent formula that uses only instances of *until* that are non-strict in the first argument. More precisely, the replacement also exploits Formula 3, which shows how to express any occurrence of the strict *nowon* operator (employed in the equivalence of Formula 6) using non-strict *nowon* (and thus, non-strict *until*) only.

Formulas 9 and 10 further show that non-punctuality of the intervals can be preserved; that is, it is possible to replace any occurrence of strict *until* with an equivalent formula that uses only instances of non-strict *until* whose intervals are punctual only if those bounding the replaced strict *until* are. Therefore, we have established the following.

Theorem 1. *For non-Zeno behaviors over dense-time domains:*

- *the language* MTL *(*MTL *with non-strict* until*) is as expressive as the language* $\widetilde{\text{MTL}}$ *(*MTL *with strict* until*);*

- *the language* MITL *is as expressive as the language* $\widetilde{\text{MITL}}$*.*

5 Extensions to Other Semantics and Languages

This section discusses how the results of Theorem 1 can be extended to some common variations of the MTL language or of its semantics. More precisely, we discuss to what extent our results apply to semantics with a mono-infinite time domain, to MTL extensions with past operators, to the well-known timed interval sequence and timed word semantics, and to generic (Zeno) behaviors.

Mono-infinite time domains. In proving the various equivalence formulas in the previous section, no assumptions were made about the dense temporal domain $\mathbb{T}$ being bi-infinite or mono-infinite. Therefore, Theorem 1 holds in particular for mono-infinite time domains that is, in practice, the sets of nonnegative rationals $\mathbb{Q}_{\geq 0}$ and nonnegative reals $\mathbb{R}_{\geq 0}$.

As we hinted at in Section 2.1.2, in the literature, whenever a mono-infinite time domain such as $\mathbb{R}_{\geq 0}$ is adopted, the interpretation relation $b \models_{\mathbb{R}_{\geq 0}} \phi$ is usually defined as $b(0) \models_{\mathbb{R}_{\geq 0}} \phi$; this is called *initial satisfiability*. On the contrary, in Section 2.1.2, we defined a different interpretation relation called *global satisfiability* and defined as $\forall t \in \mathbb{R}_{\geq 0} : b(t) \models_{\mathbb{R}_{\geq 0}} \phi$.

Our results still apply for the initial satisfiability relation. In fact, we have shown, given an $\widetilde{\text{MTL}}$ formula ϕ, how to derive an MTL formula ϕ' such that $b(t) \models_{\mathbb{T}} \phi$ if and only if $b(t) \models_{\mathbb{T}} \phi'$, for all times $t \in \mathbb{T}$. Since t is generic, in particular our results hold for $t = 0$. Therefore, we can express any $\widetilde{\text{MTL}}$ formula ϕ such that $b(0) \models_{\mathbb{R}} \phi$ with another MTL formula ϕ' such that $b(0) \models_{\mathbb{T}} \phi'$. This shows that Theorem 1 holds for mono-infinite time domains independent of whether global or initial satisfiability is chosen.

Adding past operators. A common syntactic extension of MTL is obtained by adding past modal operators. That is, we introduce a symmetric version of the bounded *until* operator, named *since* and denoted with the symbol S, that deals with past time instants. We define the following semantics, respectively for the non-strict (in the first argument) *since* S_I and for its strict counterpart $\widetilde{\mathsf{S}}_I$.

$b(t) \models_{\mathbb{T}} \mathsf{S}_I(\phi_1, \phi_2)$ iff there exists $d \in I$ such that $b(t-d) \models_{\mathbb{T}} \phi_2$ and, for all $u \in [0, d)$ it is $b(t-u) \models_{\mathbb{T}} \phi_1$

$b(t) \models_{\mathbb{T}} \widetilde{\mathsf{S}}_I(\phi_1, \phi_2)$ iff there exists $d \in I$ such that $b(t-d) \models_{\mathbb{T}} \phi_2$ and, for all $u \in (0, d)$ it is $b(t-u) \models_{\mathbb{T}} \phi_1$

We name $\mathrm{MTL}^{\mathrm{P}}$ the extension of MTL with a non-strict *since*, and $\widetilde{\mathrm{MTL}}^{\mathrm{P}}$ the extension of $\widetilde{\mathrm{MTL}}$ with a strict *since*.

It is known that, over dense time, adding past operators to MTL strictly increases its expressive power [6, 21]. However, let us illustrate how Theorem 1 can be extended to deal with $\mathrm{MTL}^{\mathrm{P}}$ and $\widetilde{\mathrm{MTL}}^{\mathrm{P}}$ as well. First of all, let us rewrite Formulas 2–10 so that they refer to the past, rather than to the future; in other words, we replace every occurrence of the *until* operator (and its specializations) with a *since*. For instance, Formula 4 becomes:

$$\widetilde{\mathsf{S}}_{\langle a,b\rangle}(\phi_1, \phi_2) \ \equiv\ \overleftarrow{\Diamond}_{\langle a,b\rangle}(\phi_2) \wedge \overleftarrow{\Box}_{(0,a]}(\mathsf{S}(\phi_1, \phi_2)) \tag{11}$$

where $\overleftarrow{\Diamond}_I(\phi) \equiv \mathsf{S}_I(\top, \phi)$ and $\overleftarrow{\Box}_I(\phi) \equiv \neg\overleftarrow{\Diamond}_I(\neg\phi)$.

Let us first assume a bi-infinite time domain such as $\mathbb{R}$. Then, it is straightforward to check that all of the proofs of Formulas 2–10 work for the new formulas with *since* in place of *until*, if we just replace, in the proofs, any distance d in the future of the current instant of t with a symmetric distance in the past of t (i.e., $-d$). Let us show this by modifying the proof of Formula 4 into that of Formula 11.

Proof of Formula (11). Let us start with the $\Rightarrow$ direction: assume that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{S}}_{\langle a,b\rangle}(\phi_1, \phi_2)$. That is, there exists a $u \in \langle t-b, t-a)$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $v \in (u, t)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. From $b(u) \models_{\mathbb{T}} \phi_2$ it follows immediately that $\overleftarrow{\Diamond}_{\langle a,b\rangle}(\phi_2)$, so the first conjunct is proved.

Then, let us show that $b(t) \models_{\mathbb{T}} \overleftarrow{\Box}_{(0,a]}(\mathsf{S}(\phi_1, \phi_2))$. Let α be any instant in $(0, a]$; we have to show that $b(t-\alpha) \models_{\mathbb{T}} \mathsf{S}(\phi_1, \phi_2)$. Notice that $\langle t-b, t-a) \subseteq (-\infty, t-\alpha)$, as $t - a \leq t - \alpha$, therefore $u \in (-\infty, t-\alpha)$ *a fortiori*. Moreover, $(u, t-\alpha] \subset (u, t)$, as $\alpha > 0$, so for all $v' \in (u, t-\alpha]$ it is $b(v') \models_{\mathbb{T}} \phi_1$. Therefore, we have shown that $b(t-\alpha) \models_{\mathbb{T}} \mathsf{S}(\phi_1, \phi_2)$.

Let us now consider the $\Leftarrow$ direction. Notice that $b(t) \models_{\mathbb{T}} \overleftarrow{\Box}_{(0,a]}(\mathsf{S}(\phi_1, \phi_2))$ implies that $b(t) \models_{\mathbb{T}} \overleftarrow{\Box}_{(0,a)}(\phi_1)$. Moreover, in particular $b(t-a) \models_{\mathbb{T}} \mathsf{S}(\phi_1, \phi_2)$. That is, there exists a $u \in (-\infty, t-a)$ such that $b(u) \models_{\mathbb{T}} \phi_2$ and, for all $v \in (u, t-a]$ it is $b(v) \models_{\mathbb{T}} \phi_1$.

Let us now consider the case $u \in \langle t-b, t-a)$. All in all, ϕ_1 holds over the interval (u, t), and ϕ_2 holds at u; therefore, we have $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{S}}_{\langle a,b\rangle}(\phi_1, \phi_2)$.

Otherwise, let us consider the case $u \notin \langle t-b, t-a)$; therefore $u \in (-\infty, t-a) \setminus \langle t-b, t-a)$. In particular, this implies that for all $v \in (t-b, t)$ it is $b(v) \models_{\mathbb{T}} \phi_1$. Moreover, we are also assuming that $b(t) \models_{\mathbb{T}} \overleftarrow{\Diamond}_{\langle a,b\rangle}(\phi_2)$ in this branch of the proof.

That is, there exists a $u' \in \langle t-b, t-a)$ such that $b(u') \models_{\mathbb{T}} \phi_2$. Since $(u', t) \subseteq (t-b, t)$, then we have shown that $b(t) \models_{\mathbb{T}} \widetilde{\mathsf{S}}_{\langle a,b \rangle}(\phi_1, \phi_2)$, as required. □

Intuitively, problems may arise with mono-infinite time domains such as $\mathbb{R}_{\geq 0}$, where time has a lower bound. For such domains, a distance $d > 0$ is defined in the past only at instants greater than or equal to d itself. However, a close scrutiny of our proofs shows the following: from any current instant t, in the proofs we never consider any instant in the future of t that is beyond some distance d, such that the existence of the instant $t + d$ is an hypothesis of the proof itself (i.e., it is asserted by an existential quantification of an *until* formula that is assumed to hold). In other words, the proofs never predicate about instants whose existence is not constructively assumed. Therefore, if we consider the same proofs "reversed" to the past, we never reference instants that are not guaranteed to exist (i.e., that fall before the minimum of the mono-infinite time domain).

As a witness to this claim, consider again the proof of Formula 11: the existence of an instant $u \in \langle t - b, t - a)$ in the past of t — such that a formula ϕ_2 holds at u — is assumed as hypothesis. Then, no instant occurring before u is ever referenced in the remainder of the proof.

All in all, we conclude that Theorem 1 holds for the past as well.

Theorem 2. *For non-Zeno behaviors over dense-time domains:*

- *the language* MTL^{P} *is as expressive as the language* $\widetilde{\mathrm{MTL}}^{\mathrm{P}}$*;*
- *the language*[4] $\mathrm{MITL}^{\mathrm{P}}$ *is as expressive as the language* $\widetilde{\mathrm{MITL}}^{\mathrm{P}}$.

Timed interval sequences. The timed interval sequence [2, 5] is a common model over which MTL formulas are interpreted. In practice, timed interval sequences are represented exactly by those behaviors b that are non-Zeno [12, 22, 16]. Therefore, Theorem 1 applies to interpretations over timed interval sequences.

Timed words. The timed word [2, 5] is another common model over which MTL formulas are interpreted. In [6, Lemma 4] it is shown how to express any timed word ρ as a timed interval sequence $\kappa(\rho)$, and how to modify any $\widetilde{\mathrm{MTL}}$ formula ϕ into another $\widetilde{\mathrm{MTL}}$ formula ϕ' such that $\rho \models_{\mathbb{T}} \phi$ if and only if $\kappa(\rho) \models_{\mathbb{T}} \phi'$. Through this simple translation process, timed words can be represented as Boolean behaviors. Therefore, given an $\widetilde{\mathrm{MTL}}$ formula ϕ interpreted over timed words, one can use Theorem 1 to derive an MTL formula ϕ' whose behaviors correspond to the timed words of ϕ.

[4] $\mathrm{MITL}^{\mathrm{P}}$ and $\widetilde{\mathrm{MITL}}^{\mathrm{P}}$ are defined in an obvious manner.

However, timed word interpretations may adopt definitions of the basic operators that differ in some aspects from ours (when interpreted "verbatim" over timed words). For instance, sometimes *weakly monotonic* timed words are used [2], where time is not required to strictly increase at every event. In such cases, different semantics for the *until* operator are typically adopted (see [15] for an example). Evaluating exactly to what extent our results extend to such variations is beyond the scope of the present paper, and it belongs to future work.

Generic (and Zeno) behaviors. In proving Theorem 1, we considered non-Zeno behaviors only. More precisely, the non-Zenoness assumption is required in the proof of Formula 3, which is then used in Formula 6.

For a Zeno behavior b and a formula ϕ, it may be that $b(t) \models_{\mathbb{T}} \neg\widetilde{\bigcirc}(\phi) \wedge \neg\widetilde{\bigcirc}(\neg\phi)$ at some instant t, contradicting Formula 1. For instance, consider a behavior b such that $\mathsf{p} \in b(u)$ iff $u = 2^{-i}$ for some $i \in \mathbb{N}$. Then, p switches from true to false infinitely often in any finite interval to the right of the origin, so its value is undefined to the strict right of the origin. In this case, Formula 3 does not hold.

Whether, over Zeno behaviors, strict operators are more expressive than non-strict ones is — to the best of our knowledge — an open problem. We conjecture that the answer is affirmative, and in particular that a non-strict *until* operator cannot be used to distinguish between Zeno and non-Zeno behaviors. We are currently working on detailing a proof of this conjecture.

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